Companion Appendix for
Optimal Sharing Strategies in Dynamic Games of Research and Development

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We fully characterize the subgame perfect equilibrium of the game outlined in the paper for the case when $N = 2$ and the firms are symmetric. We do the analysis by dividing the parameter space into regions. Section 1 is an introductory section with a diagram illustrating the regions. In Section 2, we provide the boundary conditions for the regions. Section 3 has a table that summarizes the equilibria by region. The equilibria are derived in Sections 4 and 5.

1 Introduction

There are five parameters: the discount rate is r, the flow cost of research is c , the hazard rate of a research success is α, duopoly profit is π^D, and monopoly profit is π^M. We assume that the parameters are positive and that $\pi^M > \pi^D$. Throughout the analysis, we use the notation $\widetilde{\pi}^D = \frac{\pi^D}{r}$ and $\widetilde{\pi}^M = \frac{\pi^M}{r}$. We divide the parameters into 19 different regions. The diagram below illustrates the regions when $r = 0.2, \alpha = 0.5$ and $c = 0.5$. Regions 5 and 12 do not appear on the diagram because they are empty for these values r, α and c.

2 The Regions

In this section, we provide the boundary conditions that define each of 19 regions. Regions 1, 2 and 3 are subregions of what we define as Region A in the paper. As we show in Section 4, no firm ever drops out of the game in Region A. The other subregions comprise Region B. As we show in Section 5, the lagging firm always drops out at the history $(2, 0; NS)$ in Region B.

Region A

Region 1:

$\pi^D > \frac{cr}{\alpha}(2 + \frac{r}{\alpha})$ and $\pi^D < \pi^M < 2\pi^D + c$

Region 2:

$\pi^D > \frac{cr}{\alpha}(2 + \frac{r}{\alpha})$ and $\pi^M > 2\pi^D + c$ and

$2\pi^D > \pi^M \frac{(2\alpha^2 - r^2)}{(3\alpha^2 + 2\alpha r)} - c\frac{(2\alpha + r)^2}{(3\alpha^2 + 2\alpha r)}$

[1]This appendix is not necessary for the main results stated in the paper. For the interested reader, we present a full characterization of the subgame perfect equilibria of the game.

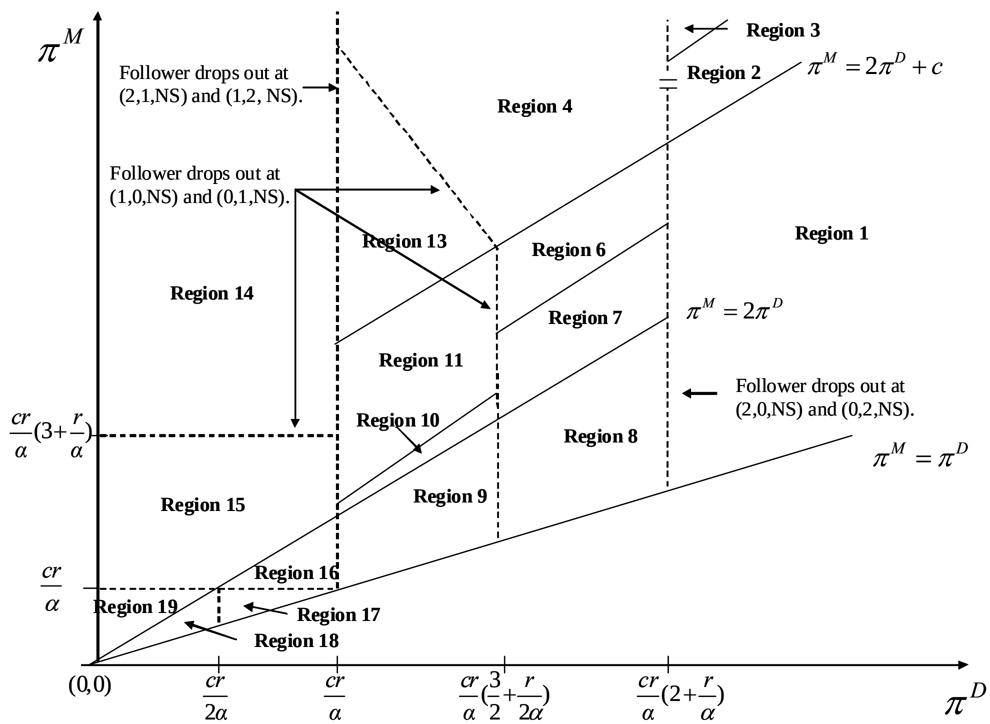

Figure 1: Regions for $\alpha = c = 0.5$ and $r = 0.2$

Region 3:
$\pi^D > \frac{cr}{\alpha}(2 + \frac{r}{\alpha})$ and $\pi^M > 2\pi^D + c$ and
$2\pi^D < \pi^M \frac{(2\alpha^2 - r^2)}{(3\alpha^2 + 2\alpha r)} - c\frac{(2\alpha+r)^2}{(3\alpha^2+2\alpha r)}$

Region B

Region 4:
$\frac{cr}{\alpha} < \pi^D < \frac{cr}{\alpha}(2 + \frac{r}{\alpha})$ and $\pi^M > 2\pi^D + c$ and
$\pi^M > -\frac{2\alpha}{r}\pi^D + (2 + \frac{r}{\alpha})^2 c$ and $\pi^D < \frac{(2\alpha - r)}{4\alpha}\pi^M$
Region 5:
$\frac{cr}{\alpha} < \pi^D < \frac{cr}{\alpha}(2 + \frac{r}{\alpha})$ and $\pi^M > 2\pi^D + c$ and
$\pi^M > -\frac{2\alpha}{r}\pi^D + (2 + \frac{r}{\alpha})^2 c$ and $\pi^D > \frac{(2\alpha - r)}{4\alpha}\pi^M$
Region 6:
$\frac{cr}{\alpha}(\frac{3}{2} + \frac{r}{2\alpha}) < \pi^D < \frac{cr}{\alpha}(2 + \frac{r}{\alpha})$ and $2\pi^D < \pi^M < 2\pi^D + c$ and
$\pi^M > \frac{4(\alpha+r)}{2\alpha+r}\pi^D + \frac{2cr}{2\alpha+r}$
Region 7:
$\frac{cr}{\alpha}(\frac{3}{2} + \frac{r}{2\alpha}) < \pi^D < \frac{cr}{\alpha}(2 + \frac{r}{\alpha})$ and $2\pi^D < \pi^M < 2\pi^D + c$ and
$\pi^M < \frac{4(\alpha+r)}{2\alpha+r}\pi^D + \frac{2cr}{2\alpha+r}$
Region 8:
$\frac{cr}{\alpha}(\frac{3}{2} + \frac{r}{2\alpha}) < \pi^D < \frac{cr}{\alpha}(2 + \frac{r}{\alpha})$ and $\pi^D < \pi^M < 2\pi^D$
Region 9:
$\frac{cr}{\alpha} < \pi^D < \frac{cr}{\alpha}(\frac{3}{2} + \frac{r}{2\alpha})$ and $\pi^D < \pi^M < 2\pi^D$
Region 10:
$\frac{cr}{\alpha} < \pi^D < \frac{cr}{\alpha}(\frac{3}{2} + \frac{r}{2\alpha})$ and $2\pi^D < \pi^M < 2\pi^D + c$ and
$\pi^M < \frac{4(\alpha+r)}{2\alpha+r}\pi^D - \frac{r^2}{\alpha(2\alpha+r)}c$
Region 11:
$\frac{cr}{\alpha} < \pi^D < \frac{cr}{\alpha}(\frac{3}{2} + \frac{r}{2\alpha})$ and $2\pi^D < \pi^M < 2\pi^D + c$ and
$\pi^M > \frac{4(\alpha+r)}{2\alpha+r}\pi^D - \frac{r^2}{\alpha(2\alpha+r)}c$
Region 12:
$\frac{cr}{\alpha} < \pi^D < \frac{cr}{\alpha}(2 + \frac{r}{\alpha})$ and $\pi^M > 2\pi^D + c$ and
$\pi^M < -\frac{2\alpha}{r}\pi^D + (2 + \frac{r}{\alpha})^2 c$ and $4\alpha\pi^D > (2\alpha - r)\,\pi^M + cr\left(2 + \frac{r}{\alpha}\right)$
Region 13:
$\frac{cr}{\alpha} < \pi^D < \frac{cr}{\alpha}(2 + \frac{r}{\alpha})$ and $\pi^M > 2\pi^D + c$ and
$\pi^M < -\frac{2\alpha}{r}\pi^D + (2 + \frac{r}{\alpha})^2 c$ and $4\alpha\pi^D < (2\alpha - r)\,\pi^M + cr\left(2 + \frac{r}{\alpha}\right)$
Region 14:
$\pi^D < \frac{cr}{\alpha}$ and $\pi^M > 2\pi^D$ and $\pi^M > \frac{cr}{\alpha}\left(3 + \frac{r}{\alpha}\right)$
Region 15:
$\pi^D < \frac{cr}{\alpha}$ and $\pi^M > 2\pi^D$ and $\frac{cr}{\alpha} < \pi^M < \frac{cr}{\alpha}\left(3 + \frac{r}{\alpha}\right)$
Region 16:

$\frac{cr}{2\alpha} < \pi^D < \frac{cr}{\alpha}$ and $\pi^D < \pi^M < 2\pi^D$ and $\frac{cr}{\alpha} < \pi^M < \frac{cr}{\alpha}(2 + \frac{r}{\alpha})$

Region 17:

$\frac{cr}{2\alpha} < \pi^D < \frac{cr}{\alpha}$ and $\pi^D < \pi^M < 2\pi^D$ and $\pi^M < \frac{cr}{\alpha}$

Region 18:

$\pi^D < \frac{cr}{2\alpha}$ and $\pi^D < \pi^M < 2\pi^D$ and $\pi^M < \frac{cr}{\alpha}$

Region 19:

$\pi^D < \frac{cr}{2\alpha}$ and $\pi^M > 2\pi^D$ and $\pi^M < \frac{cr}{\alpha}$

3 The Equilibria

Here, we summarize the equilibria by region. The derivations of the equilibria follow in Sections 4 and 5. For symmetric histories such as $(2, 1)$ and $(1, 2)$, we analyze only one of the histories as the analysis is the same for both. Some of the regions have subregions with distinct equilibrium actions. The definitions of the subregions are given in Sections 4 and 5 as part of the equilibrium derivation. In some of the equilibria, a firm is indifferent between two actions such as S or NS. In most cases, the choice of action does not affect the firm's decisions earlier in the game. If it does, we indicate that. The non-monotonic equilibria appear in Regions 6, 11, 17, and 18. For these equilibria, the firms share step 2 at $(2, 1)$, but do not share step 1 at $(1, 0)$. In Region 6, this non-monotonic behavior occurs along the equilibrium path. In the other regions, the behavior occurs off the equilibrium path.

4 Derivation of the Equilibria in Region A

We derive the equilibria in Region A. As we show, no firm ever drops out of the game in Region A. The firms incentives to share research decline over time: that is, the equilibria satisfy the monotonicity property. The monotonicity property for $N = 2$ states that if the firms share step 2 at $(2, 1)$, then they also share step 1 at $(1, 0)$.

4.1 Region 1 (monotonicity property holds)

We determine the equilibrium strategies by using backward induction. **At** $(2, 2)$**, each firm produces output** and earns discounted duopoly profits of:

$$V_1(2,2) = V_2(2,2) = \widetilde{\pi}^D = \frac{\pi^D}{r}. \quad (1)$$

At $(2, 1; NS)$**, firm 1 is finished with its research and produces output**. Firm 2 invests if

$$V_2(2,1;NS) = \frac{\alpha V_2(2,2) - c}{\alpha + r} = \frac{\alpha \widetilde{\pi}^D - c}{\alpha + r} > 0 \quad (2)$$

or

$$\pi^D > \frac{cr}{\alpha}. \tag{3}$$

This condition holds in Region 1 so that **firm 2 invests at** $(2,1;NS)$.

The firms share at $(2,1)$ iff this maximizes their joint profits. Their joint profits under sharing are

$$V_J(2,2) = V_1(2,2) + V_2(2,2) = 2\widetilde{\pi}^D = \frac{2\pi^D}{r}$$

since when the firms share, the game reaches the history $(2,2)$. Joint profits under no sharing are

$$V_J(2,1;NS) = V_1(2,1;NS) + V_2(2,1;NS) = \frac{\pi^M + 2\alpha\widetilde{\pi}^D - c}{\alpha + r} \tag{4}$$

where

$$V_1(2,1;NS) = \frac{\pi^M + \alpha V_1(2,2)}{\alpha + r} = \frac{\pi^M + \alpha\widetilde{\pi}^D}{\alpha + r}$$

and $V_2(2,1;NS)$ is as defined in (2). We get $S \succ NS \iff$

$$\begin{aligned} 2\widetilde{\pi}^D(\alpha + r) &> \pi^M + 2\alpha\widetilde{\pi}^D - c \\ 2\pi^D + c &> \pi^M. \end{aligned} \tag{5}$$

This condition holds in Region 1 and **the firms share step 2 at** $(2,1)$.

At $(1,1)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(1,1) = \frac{\alpha V_2(1,2) + \alpha V_2(2,1) - c}{2\alpha + r} = \frac{\alpha V_J(2,1) - c}{2\alpha + r} > 0. \tag{6}$$

Since the firms share at $(2,1)$, $V_J(2,1) = 2\widetilde{\pi}^D$. Substituting, we get

$$V_2(1,1) = \frac{2\alpha\widetilde{\pi}^D - c}{2\alpha + r} > 0. \tag{7}$$

This simplifies to

$$\pi^D > \frac{cr}{2\alpha}.$$

This holds in Region 1 by assumption. Hence, each firm invests at $(1,1)$ if the other does.

If firm 1 does not invest at $(1,1)$, the new history is $(X,1)$. Firm 2 invests if

$$V_2(X,1) = \frac{\alpha V_2(X,2) - c}{\alpha + r} = \frac{\alpha\widetilde{\pi}^M - c}{\alpha + r} > 0, \tag{8}$$

where the last equality holds because **at** $(X,2)$, **firm** 2 **produces output** and earns discounted monopoly profits of $\widetilde{\pi}^M = \frac{\pi^M}{r}$. The condition simplifies to

$$\pi^M > \frac{cr}{\alpha}. \tag{9}$$

This condition holds in Region 1 since $\pi^M > \pi^D$ and $\pi^D > \frac{cr}{\alpha}$ by assumption. Hence, **firm 2 invests at** $(X,1)$. It follows that **both firms invest at** $(1,1)$.

At $(2,0;NS)$**, firm 1 produces output.** Firm 2 stays in the race if

$$V_2(2,0;NS) = \frac{\alpha V_2(2,1) - c}{\alpha + r} > 0.$$

Since the lagging firm has no bargaining power, its earnings under sharing are the same as its earnings under no sharing at the history $(2,1)$. The earnings under no sharing, $V_2(2,1;NS)$, are given in (2). Substituting and rearranging gives us

$$V_2(2,0;NS) = \frac{\alpha^2 \widetilde{\pi}^D - c(2\alpha + r)}{(\alpha + r)^2} > 0 \tag{10}$$

which simplifies to

$$\pi^D > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right). \tag{11}$$

In Region 1 this condition holds by assumption and **the lagging firm stays in the race at** $(2,0;NS)$.

To see whether the firms share step 1 at $(2,0)$, we compare joint profits under sharing with joint profits under no sharing. Joint profits under sharing are $V_J(2,2) = 2\widetilde{\pi}^D$ since if the firms share, the game reaches the history $(2,1)$ and we know from condition (5) that at this history the firms share. Joint profits under no sharing are

$$V_J(2,0;NS) = \frac{\alpha V_2(2,1) - c}{\alpha + r} + \frac{\pi^M + \alpha V_1(2,1)}{\alpha + r} = \frac{\pi^M + \alpha V_J(2,1) - c}{\alpha + r}.$$

Since the firms share at $(2,1)$, $V_J(2,1) = 2\widetilde{\pi}^D$. Substituting we get $S \succ NS \iff$

$$\begin{aligned} 2\widetilde{\pi}^D(\alpha + r) &> \pi^M + 2\alpha\widetilde{\pi}^D - c \\ 2\pi^D + c &> \pi^M. \end{aligned}$$

This condition holds in Region 1, and hence **the firms share step 1 at** $(2,0)$.

At $(1,0;NS)$, firm 2 invests if

$$V_2(1,0;NS) = \frac{\alpha V_2(1,1) + \alpha V_2(2,0) - c}{2\alpha + r} > 0.$$

We can substitute for $V_2(1,1)$ from (7). Moreover, since the lagging firm is assumed to have no bargaining power, it makes the same profit at $(2,0;NS)$ as at $(2,0)$ even though the firms share. Substituting for $V_2(2,0) = V_2(2,0;NS)$ from (10) and simplifying gives

$$\widetilde{\pi}^D > \frac{c}{\alpha}\left[\frac{\alpha(2\alpha+r)^2 + (3\alpha+r)(\alpha+r)^2}{2\alpha(\alpha+r)^2 + (2\alpha+r)\alpha^2}\right].$$

By assumption, $\widetilde{\pi}^D > \frac{c}{\alpha}\left(2+\frac{r}{\alpha}\right)$ in Region 1. Hence, the above condition holds if

$$\left(2+\frac{r}{\alpha}\right) > \left[\frac{\alpha(2\alpha+r)^2 + (3\alpha+r)(\alpha+r)^2}{2\alpha(\alpha+r)^2 + (2\alpha+r)\alpha^2}\right].$$

This simplifies to

$$\alpha(\alpha+r) > 0.$$

Since $\alpha, r > 0$, this condition holds and **the lagging firm stays in the race at** $(1,0;NS)$. It is straightforward to show that **the leading firm stays in the race at** $(1,0;NS)$.[2]

To see whether the firms share step 1 at $(1,0)$, we compare joint profits under sharing with joint profits under no sharing. If the firms share, the game reaches the history $(1,1)$. Hence, joint profits under sharing are $V_J(1,1)$. Joint profits under no sharing are

$$V_J(1,0;NS) = \frac{\alpha V_J(2,0) + \alpha V_J(1,1) - 2c}{2\alpha+r} = \frac{2\alpha\widetilde{\pi}^D + \alpha V_J(1,1) - 2c}{2\alpha+r}.$$

We have $S \succ NS \iff$

$$(2\alpha+r)V_J(1,1) > 2\alpha\widetilde{\pi}^D + \alpha V_J(1,1) - 2c$$

Substituting for $V_J(1,1) = 2V_2(1,1)$ from equation (7), we get

$$\begin{aligned}(\alpha+r)\left(4\alpha\widetilde{\pi}^D - 2c\right) &> (2\alpha+r)\left(2\alpha\widetilde{\pi}^D - 2c\right)\\ \pi^D &> -c,\end{aligned}$$

which is trivially true. Hence, **the firms share step 1 at** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0) + \alpha V_2(0,1) - c}{2\alpha+r} > 0.$$

[2] For an intuitive explanation, first consider the case when the firms never share in the future. Since the leading firm has a shorter expected time to completion, it would have higher incentives to stay in the race than the lagging firm. Now consider the case when the firms share at least once in the future. Since the leading firm has full bargaining power, it would again prefer to stay in the race whenever the lagging firm finds it profitable to do so.

Since the firms share step 1 at $(1,0)$, this condition becomes

$$\frac{\alpha V_J(1,1) - c}{2\alpha + r} > 0.$$

Substituting for $V_J(1,1) = 2V_2(1,1)$ from equation (7) and simplifying, we get

$$\widetilde{\pi}^D > \frac{c}{\alpha}\left(1 + \frac{r}{4\alpha}\right).$$

This condition holds in Region 1 due to condition (11). Hence, each firm invests at $(0,0)$ if the other firm does.

Assuming firm 1 does not invest, firm 2 invests at $(X,0)$ if

$$V_2(X,0) = \frac{\alpha V_2(X,1) - c}{\alpha + r} = \frac{\alpha\left(\frac{\alpha\widetilde{\pi}^M - c}{\alpha + r}\right) - c}{\alpha + r} > 0 \tag{12}$$

where we substituted for $V_2(X,1)$ using (8). Simplifying we get

$$\pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right) \tag{13}$$

which holds in Region 1 by condition (11) and the assumption that $\pi^M > \pi^D$. Hence, **firm 2 invests at** $(X,0)$. It follows that **both firms invest at** $(0,0)$.

4.2 Region 2 (monotonicity property holds)

In Region 2, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(X,2)$, $(X,1)$ and $(X,0)$ is the same as in Region 1 and we do not repeat it here. At $(2,2)$ both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 invests. At $(X,1)$ and $(X,0)$, firm 2 invests.

At the history $(2,1)$, we know from the analysis in Region 1 that the firms share if condition (5) holds. In Region 2, this condition is violated as we assume that

$$2\pi^D + c < \pi^M.$$

Hence, **the firms do not share at** $(2,1)$.

Working back, we get to the history $(1,1)$. Assuming firm 1 invests, firm 2 will also invest if

$$V_2(1,1) = \frac{\alpha V_2(1,2) + \alpha V_2(2,1) - c}{2\alpha + r} = \frac{\alpha V_J(2,1) - c}{2\alpha + r} > 0. \tag{14}$$

Since the firms do not share at $(2,1)$, we can substitute for $V_J(2,1) = V_J(2,1;NS)$ from (4). Simplifying, we get

$$V_2(1,1) = \frac{\alpha(\pi^M + 2\alpha\widetilde{\pi}^D) - c(2\alpha + r)}{(2\alpha + r)(\alpha + r)} > 0 \tag{15}$$

which simplifies further to

$$\pi^M + \frac{2\alpha}{r}\pi^D > c\left(2 + \frac{r}{\alpha}\right). \tag{16}$$

This holds in Region 2, because by assumption $\pi^M > 2\pi^D$ and $\pi^D > 2\frac{cr}{\alpha}$. Thus each firm invests at $(1, 1)$ if the other firm invests. From the analysis of Region 1, if firm 1 does not invest, the new history is $(X, 1)$ and firm 2 invests. It follows that **both firms invest at** $(1, 1)$**.**

At $(2, 0; NS)$**, firm 1 produces output.** Firm 2 stays in the race under no sharing if

$$V_2(2, 0; NS) = \frac{\alpha V_2(2, 1) - c}{\alpha + r} > 0. \tag{17}$$

Since the firms do not share at $(2, 1)$ and the lagging firm stays in the race,

$$V_2(2, 1) = \left(\frac{\alpha\widetilde{\pi}^D - c}{\alpha + r}\right).$$

Using this, we get

$$V_2(2, 0; NS) = \frac{\alpha^2\widetilde{\pi}^D - c(2\alpha + r)}{(\alpha + r)^2} > 0 \tag{18}$$

which simplifies to

$$\pi^D > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right),$$

which is the same condition as (11). The conditions are the same because the lagging firm has no bargaining power. The condition holds in Region 2 by assumption and **firm 2 stays in the race at** $(2, 0; NS)$.

To see whether the firms share step 1 at $(2, 0)$, we compare joint profits under sharing with joint profits under no sharing. If the firms share step 1, their joint profits are given by $V_J(2, 1)$. Joint profits under no sharing are

$$\begin{aligned} V_J(2, 0; NS) &= V_1(2, 0; NS) + V_2(2, 0; NS) \\ &= \frac{\pi^M + \alpha V_1(2, 1)}{\alpha + r} + \frac{\alpha V_2(2, 1) - c}{\alpha + r} \\ &= \frac{\pi^M + \alpha V_J(2, 1) - c}{\alpha + r}. \end{aligned}$$

Combining we get $S \succ NS \iff$

$$V_J(2, 1) > \frac{\pi^M + \alpha V_J(2, 1) - c}{\alpha + r}$$

Since the firms do not share at $(2,1)$, we can substitute for $V_J(2,1)$ from (4). Simplifying, we get

$$\begin{aligned}\left[\frac{\pi^M + 2\alpha\widetilde{\pi}^D - c}{\alpha + r}\right] r &> \pi^M - c \\ 2\pi^D + c &> \pi^M.\end{aligned}$$

This condition is violated in Region 2, so **the firms do not share step 1 at** $(2,0)$.

At $(1,0;NS)$, firm 2 invests if

$$V_2(1,0;NS) = \frac{\alpha V_2(1,1) + \alpha V_2(2,0) - c}{2\alpha + r} > 0.$$

Substituting for $V_2(1,1)$ and $V_2(2,0)$ from (14) and (17) respectively and rearranging, we get

$$\frac{\alpha V_J(2,1) - c}{2\alpha + r} + \frac{\alpha V_2(2,1) - c}{\alpha + r} > \frac{c}{\alpha}$$

Since the lagging firm chooses to stay in at $(2,0;NS)$, we know that

$$\frac{\alpha V_2(2,1) - c}{\alpha + r} > 0.$$

Hence, a sufficient condition for firm 2 to invest at $(1,0;NS)$ is

$$\frac{\alpha V_J(2,1) - c}{2\alpha + r} > \frac{c}{\alpha}.$$

As above, we can substitute for $V_J(2,1)$ from (4). Simplifying gives us

$$\pi^M + 2\alpha\widetilde{\pi}^D > \frac{c}{\alpha}\left(4\alpha + 4r + \frac{r^2}{\alpha}\right).$$

Since $\pi^M > \pi^D$, this condition holds if

$$\widetilde{\pi}^D(r + 2\alpha) > \frac{c}{\alpha}\left(4\alpha + 4r + \frac{r^2}{\alpha}\right).$$

Given condition (11), which holds in Region 2, this condition holds and **the lagging firm stays in at the history** $(1,0;NS)$. It is straightforward to show that **the leading firm stays in at** $(1,0;NS)$.

To see whether the firms share step 1 at $(1,0)$, we compare joint profits under sharing with joint profits under no sharing. Joint profits under sharing are $V_J(1,1)$. Joint profits under no sharing are

$$V_J(1,0;NS) = \frac{\alpha V_J(2,0) + \alpha V_J(1,1) - 2c}{2\alpha + r}. \tag{19}$$

We get $S \succ NS \iff$

$$(\alpha + r) V_J(1,1) > \alpha V_J(2,0) - 2c.$$

We can substitute for $V_J(1,1) = 2V_2(1,1)$ using (15). Since there is no sharing at either $(2,0)$ or $(2,1)$, we use equation (4) to get

$$V_J(2,0) = \frac{\pi^M + \alpha V_J(2,1) - c}{\alpha + r} = \frac{(2\alpha + r)\pi^M + 2\alpha^2 \widetilde{\pi}^D - c(2\alpha + r)}{(\alpha + r)^2}.$$

Substituting and simplifying we get $S \succ NS \iff$

$$2\pi^D + c\frac{(2\alpha + r)^2}{(3\alpha^2 + 2\alpha r)} > \pi^M \frac{(2\alpha^2 - r^2)}{(3\alpha^2 + 2\alpha r)}. \tag{20}$$

This condition holds in Region 2, and **the firms share step 1 at** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0) + \alpha V_2(0,1) - c}{2\alpha + r} = \frac{\alpha V_J(1,0) - c}{2\alpha + r} = \frac{\alpha V_J(1,1) - c}{2\alpha + r} > 0.$$

After substituting for $V_J(1,1) = 2V_2(1,1)$ from (15), this condition simplifies to

$$\pi^M + 2\alpha\widetilde{\pi}^D > \frac{c}{2\alpha^2}(3\alpha + r)(2\alpha + r).$$

Since $\pi^M > \pi^D$, the condition holds if

$$\begin{aligned} \widetilde{\pi}^D(2\alpha + r) &> \frac{c}{2\alpha^2}(3\alpha + r)(2\alpha + r) \\ \pi^D &> \frac{cr}{\alpha}\left(\frac{3}{2} + \frac{r}{2\alpha}\right) \end{aligned}$$

This condition holds in Region 2 because by assumption

$$\pi^D > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right).$$

Hence, each firm invests at $(0,0)$ if the other firm does. We already know that firm 2 invests at $(X,0)$. It follows that **both firms invest at** $(0,0)$.

4.3 Region 3 (monotonicity property holds)

In Region 3, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(2,1)$, $(1,1)$, $(2,0;NS)$, $(2,0)$, $(1,0;NS)$, $(X,2)$, $(X,1)$ and $(X,0)$ is the same as in Region 2, and we do not repeat here. At $(2,2)$ both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 invests. At $(2,1)$, the firms do not share. At $(1,1)$, both firms invest. At $(2,0;NS)$, firm 1 produces output and firm 2 invests. At $(2,0)$, the firms do not share step 1. At $(1,0;NS)$, both firms invest. At $(X,1)$ and at $(X,0)$, firm 2 invests.

From the analysis above, we know that in Region 2 firms share at $(1,0)$ if condition (20) holds. In Region 3, the condition does not hold and we have that

$$2\pi^D + c\frac{(2\alpha+r)^2}{(3\alpha^2+2\alpha r)} < \pi^M \frac{\left(2\alpha^2 - r^2\right)}{(3\alpha^2+2\alpha r)}.$$

Hence, **the firms do not share at** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0) + \alpha V_2(0,1) - c}{2\alpha + r} = \frac{\alpha V_J(1,0) - c}{2\alpha + r} > 0. \tag{21}$$

From equation (19) we have

$$V_J(1,0) = \frac{\alpha V_J(2,0) + \alpha V_J(1,1) - 2c}{2\alpha + r}.$$

Substituting into condition (21) and simplifying, we get

$$V_J(2,0) + V_J(1,1) > \frac{c}{\alpha^2}(4\alpha + r)$$

At $(2,0)$, the firms do not share and the lagging firm stays in. At $(1,1)$, both firms stay in. Hence, we can write the condition as

$$\frac{\pi^M + \alpha V_J(2,1) - c}{\alpha + r} + \frac{2\alpha V_J(2,1) - 2c}{2\alpha + r} > \frac{c}{\alpha^2}(4\alpha + r).$$

At the history $(2,1)$ the firms do not share, and the lagging firm stays in. Substituting for $V_J(2,1) = V_J(2,1;NS)$ using (4), it is straightforward to show that this condition holds. Hence, each firm invests at $(0,0)$ if the other firm does. We already know that firm 2 invests at $(X,0)$. It follows that **both firms invest at** $(0,0)$.

5 Derivation of the Equilibria in Region B

We derive the equilibria in Region B. In Region B, the lagging firm drops out at the history $(2,0;NS)$. We show that the monotonicity property fails to hold in subregions 6, 11, 18, and 19 of Region B. In these regions, there are equilibria such that the firms share step 2 at $(2,1)$, even though they do not share step 1 at $(1,0)$. In Region 6, this non-monotonic behavior occurs along the equilibrium path. In the other regions, the behavior occurs off the equilibrium path. The equilibrium in Region 6 is derived both here and in our paper.

5.1 Region 4 (monotonicity property holds)

In Region 4, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(2,1)$, $(1,1)$, $(X,2)$ and $(X,1)$ is the same as in Region 2, and we do not repeat it here. At $(2,2)$ both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 invests. At $(2,1)$, the firms do not share step 2. At $(1,1)$ both firms invest, and at $(X,1)$ firm 2 invests.

At $(2,0;NS)$, **firm 1 produces output.** In Region 2 the lagging firm stays in the race at $(2,0;NS)$ because condition (11) holds. In Region 4, this condition is violated and we have

$$\pi^D < \frac{cr}{\alpha}\left(2+\frac{r}{\alpha}\right). \tag{22}$$

Hence, **the lagging firm exits at** $(2,0;NS)$.

To see whether the firms share step 1 at $(2,0)$, we compare joint profits under sharing with joint profits under no sharing. If the firms share step 1, the game reaches the history $(2,1)$. Since the firms do not share step 2 at $(2,1)$, joint profits under sharing step 1 at $(2,0)$ are

$$V_J\left(2,1;NS\right) = \frac{\pi^M + 2\alpha\widetilde{\pi}^D - c}{\alpha+r}.$$

Joint profits under no sharing are $V_J\left(2,0;NS\right) = \widetilde{\pi}^M$ since the lagging firm drops out at $(2,0;NS)$.

We have $S \succ NS \iff$

$$\frac{\pi^M + 2\alpha\widetilde{\pi}^D - c}{\alpha+r} > \widetilde{\pi}^M.$$

Simplifying and using $\widetilde{\pi}^M = \frac{\pi^M}{r}$ and $\widetilde{\pi}^D = \frac{\pi^D}{r}$, we get

$$2\pi^D - \frac{cr}{\alpha} > \pi^M.$$

This condition is violated in Region 4 since by assumption

$$2\pi^D + c < \pi^M.$$

Hence, **the firms do not share at the history** $(2,0)$ in Region 4. The lagging firm then exits the race.

At the history $(1,0;NS)$ the lagging firm invests if

$$V_2\left(1,0;NS\right) = \frac{\alpha V_2\left(2,0\right) + \alpha V_2\left(1,1\right) - c}{2\alpha+r} > 0.$$

Since the firms do not share at $(2,0)$ and the lagging firm exits at $(2,0;NS)$, we have $V_2\left(2,0\right)=0$. Hence, the lagging firm invests if

$$V_2\left(1,1\right) > \frac{c}{\alpha}.$$

Substituting for $V_2(1,1)$ from (15) we get

$$\pi^M + 2\alpha\widetilde{\pi}^D > c\left(2+\frac{r}{\alpha}\right)^2. \tag{23}$$

This condition holds in Region 4 by assumption, so **the lagging firm stays in at** $(1,0;NS)$. It is straightforward to show that **the leading firm stays in at** $(1,0;NS)$**.**

To see whether the firms share step 1 at $(1,0)$, we compare joint profits under sharing with joint profits under no sharing. Joint profits under sharing are $V_J(1,1)$. Joint profits under no sharing are

$$V_J(1,0;NS) = \frac{\alpha V_J(2,0) + \alpha V_J(1,1) - 2c}{2\alpha + r}. \tag{24}$$

Here $V_J(2,0) = \widetilde{\pi}^M$ since the firms do not share at $(2,0)$ and the lagging firm exits. We have $S \succ NS \Longleftrightarrow$

$$(\alpha + r)V_J(1,1) > \alpha\widetilde{\pi}^M - 2c.$$

Substituting for $V_J(1,1) = 2V_2(1,1)$ from (15) and rearranging,[3] we get

$$\pi^D > \frac{(2\alpha - r)}{4\alpha}\pi^M \tag{25}$$

This condition fails in Region 4 by assumption, and **the firms do not share at the history** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 also invests if

$$V_2(0,0) = \frac{\alpha V_2(1,0) + \alpha V_2(0,1) - c}{2\alpha + r} = \frac{\alpha V_J(1,0) - c}{2\alpha + r} > 0.$$

Since the firms do not share at $(1,0)$, we can substitute for $V_J(1,0) = V_J(1,0;NS)$ from equation (24). We get

$$\begin{aligned} \frac{\alpha V_J(2,0) + \alpha V_J(1,1) - 2c}{2\alpha + r} &> \frac{c}{\alpha} \\ V_J(2,0) + V_J(1,1) &> \frac{c}{\alpha}\left(4 + \frac{r}{\alpha}\right). \end{aligned}$$

Since $V_J(2,0) = \widetilde{\pi}^M$ and both firms invest at $(1,1)$, we can write this condition as

$$\widetilde{\pi}^M + \frac{2\alpha V_J(2,1) - 2c}{2\alpha + r} > \frac{c}{\alpha}\left(4 + \frac{r}{\alpha}\right).$$

[3] In rearranging, we do not divide any term by a potentially negative quantity.

Since the firms do not share at $(2,1)$, we can substitute for $V_J(2,1) = V_J(2,1;NS)$ from equation (4). After simplifying, it is straightforward to show that the condition holds. Hence, each firm invests at $(0,0)$ if the other firm does.

Assuming firm 1 does not invest, firm 2 will invest at $(X,0)$ if $V_2(X,0) > 0$. From the analysis of Region 1, $V_2(X,0) > 0$ if (13) holds. That is, $V_2(X,0) > 0$ if

$$\pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right).$$

Combining condition (22) and condition (23), we can show that this holds in Region 4. Hence, **firm 2 invests at** $(X,0)$ **in Region 4**.

We conclude that **both firms invest at** $(0,0)$**.**

5.2 Region 5 (monotonicity property holds)

In Region 5, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(2,1)$, $(1,1)$, $(2,0;NS)$, $(2,0)$, $(1,0;NS)$, $(X,2)$, $(X,1)$ and $(X,0)$ is the same as in Region 4, and we do not repeat it here. At $(2,2)$ both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 invests. At $(2,1)$, the firms do not share step 2. At $(1,1)$, both firms invest. At $(2,0;NS)$, firm 1 produces output and firm 2 exits the race. At $(2,0)$, the firms do not share. At $(1,0;NS)$, both firms invest. At $(X,1)$ and $(X,0)$, firm 2 invests.

From the analysis in Region 4 we know that the firms share at the history $(1,0)$ if condition (25) holds. In Region 5, condition (25) holds by assumption, and **the firms share step 1 at** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(0,1) + \alpha V_2(1,0) - c}{2\alpha + r} = \frac{\alpha V_J(1,0) - c}{2\alpha + r} = \frac{\alpha V_J(1,1) - c}{2\alpha + r} > 0,$$

where $V_J(1,0) = V_J(1,1)$ because the firms share at $(1,0)$. Substituting for $V_J(1,1) = 2V_2(1,1)$ from (15) we get

$$\pi^M + 2\alpha\widetilde{\pi}^D > \frac{c}{2\alpha^2}(3\alpha + r)(2\alpha + r). \tag{26}$$

This condition is implied by

$$\pi^M + 2\alpha\widetilde{\pi}^D > c\left(2 + \frac{r}{\alpha}\right)^2$$

which holds in Region 5 by assumption.

Hence, each firm invests at $(0,0)$ if the other firm does. As in Region 4, firm 2 invests at $(X,0)$. It follows that **both firms invest at** $(0,0)$.

5.3 Region 6 (monotonicity property fails on the equilibrium path)

In Region 6, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(2,1)$, $(1,1)$, $(X,2)$ and $(X,1)$ is the same as in Region 1, and we do not repeat it here. At $(2,2)$ both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 invests. At $(2,1)$, the firms share step 2, and both firms invest at $(1,1)$. At $(X,1)$, firm 2 invests.[4]

At $(2,0;NS)$, **firm 1 produces output.** We know from the analysis of Region 1 that firm 2 stays in the race at $(2,0;NS)$ if condition (11) holds. In Region 6 this condition is violated as we have

$$\pi^D < \frac{cr}{\alpha}\left(2+\frac{r}{\alpha}\right). \tag{27}$$

Hence, **the lagging firm exits at** $(2,0;NS)$**.**

To see whether the firms share step 1 at $(2,0)$, we compare joint profits under sharing with joint profits under no sharing. Joint profits under sharing are $V_J(2,2) = 2\widetilde{\pi}^D$ since if the firms share, the game reaches the history $(2,1)$ and we know from (5) that at this history the firms share. Joint profits under no sharing are

$$V_J(2,0;NS) = \widetilde{\pi}^M$$

since the lagging firm exits at $(2,0;NS)$. We get

$$NS \succ S \Longleftrightarrow \pi^M > 2\pi^D, \tag{28}$$

which holds in Region 6 by assumption. Hence, in Region 6, **the firms do not share at** $(2,0)$ and the lagging firm exits at $(2,0;NS)$.

At the history $(1,0;NS)$ the lagging firm invests if

$$V_2(1,0;NS) = \frac{\alpha V_2(1,1) + \alpha V_2(2,0) - c}{2\alpha + r} > 0.$$

Since the firms do not share at $(2,0)$ and the lagging firm exits at $(2,0;NS)$, we have $V_2(2,0) = 0$, and this condition simplifies to

$$V_2(1,1) > \frac{c}{\alpha}.$$

Substituting for $V_2(1,1)$ from (7), we get

$$\begin{aligned} \frac{2\alpha\widetilde{\pi}^D - c}{2\alpha + r} &> \frac{c}{\alpha} \\ \pi^D &> \frac{cr}{\alpha}\left(\frac{3}{2}+\frac{r}{2\alpha}\right). \end{aligned} \tag{29}$$

[4] Region 6 is non-empty for some values of π^M and π^D if and only if $\frac{r}{\alpha} < \frac{1}{2}(\sqrt{5}-1)$. An explanation is available on request.

This condition holds in Region 6 by assumption, and **the lagging firm stays in the race at** $(1,0;NS)$. It is straightforward to show that **the leading firm stays in at** $(1,0;NS)$**.**

To see whether the firms share step 1 at $(1,0)$, we compare joint profits under sharing with joint profits under no sharing. Joint profits under sharing are $V_J(1,1)$. Joint profits under no sharing are

$$V_J(1,0;NS) = \frac{\alpha V_J(1,1) + \alpha V_J(2,0) - 2c}{2\alpha + r}. \quad (30)$$

Since the firms do not share at $(2,0)$ and the lagging firm exits at $(2,0;NS)$, $V_J(2,0) = \widetilde{\pi}^M$. We have $NS \succ S \Longleftrightarrow$

$$\alpha\widetilde{\pi}^M - 2c > V_J(1,1)(\alpha + r)$$

Substituting for $V_J(1,1) = 2V_2(1,1)$ from (7), we get

$$\begin{aligned} \left(\alpha\widetilde{\pi}^M - 2c\right)(2\alpha + r) &> \left(4\alpha\widetilde{\pi}^D - 2c\right)(\alpha + r) \\ \pi^M &> \frac{4(\alpha + r)\pi^D}{2\alpha + r} + \frac{2cr}{2\alpha + r}. \quad (31) \end{aligned}$$

In Region 6, this condition holds by assumption and **the firms do not share at** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(0,1) + \alpha V_2(1,0) - c}{2\alpha + r} = \frac{\alpha V_J(1,0) - c}{2\alpha + r} > 0.$$

We substitute for $V_J(1,0)$ using (30), where within (30) we substitute for $V_J(1,1)$ and $V_J(2,0)$ as described above. Simplifying we get

$$(2\alpha + r)\pi^M + 4\alpha\pi^D > \frac{cr}{\alpha^2}(4\alpha + r)(2\alpha + r) + 2cr$$

Since $\pi^M > 2\pi^D$ in Region 6, the condition will hold if

$$\begin{aligned} \pi^D(8\alpha + 2r) &> \frac{cr}{\alpha^2}(4\alpha + r)(2\alpha + r) + 2cr \\ \pi^D &> \frac{cr}{2\alpha^2}\left[\frac{10\alpha^2 + 6\alpha r + r^2}{(4\alpha + r)}\right]. \end{aligned}$$

From condition (29) we know that

$$\pi^D > \frac{cr}{2\alpha^2}(3\alpha + r).$$

Hence, it is sufficient to show that

$$(3\alpha + r) > \frac{10\alpha^2 + 6\alpha r + r^2}{(4\alpha + r)}$$

if we would like to show that firm 2 will invest assuming firm 1 invests. Simplifying we get

$$2\alpha (2\alpha + r) > 0$$

which holds for all $\alpha, r > 0$. Hence, firm 2 will invest if firm 1 invests.

Assuming firm 1 does not invest, firm 2 will invest at $(X, 0)$ if $V_2 (X, 0) > 0$. From the analysis of Region 1, $V_2 (X, 0) > 0$ if (13) holds. That is, $V_2 (X, 0) > 0$ if

$$\pi^M > \frac{cr}{\alpha} \left(2 + \frac{r}{\alpha}\right).$$

In Region 6, we have that

$$\pi^M > 2\pi^D \text{ and } \pi^D > \frac{cr}{\alpha} \left(\frac{3}{2} + \frac{r}{2\alpha}\right).$$

These two conditions imply that (13) holds. Hence, **firm** 2 **invests at** $(X, 0)$. It follows that **both firms invest at** $(0, 0)$.

5.4 Region 7 (monotonicity property holds)

In Region 7 the analysis for the histories $(2, 2), (2, 1; NS), (2, 1), (1, 1), (2, 0; NS)$, $(2, 0)$, $(1, 0; NS)$, $(X, 2), (X, 1)$ and $(X, 0)$ is the same as in Region 6, and we do not repeat it here. At $(2, 2)$ both firms produce output, and at $(X, 2)$ firm 2 produces output. The firms share step 2 at $(2, 1)$. At $(2, 1; NS)$, firm 1 produces output and firm 2 invests. At $(1, 1)$, both firms invest. At $(2, 0; NS)$, firm 1 produces output and firm 2 exits the race. At $(2, 0)$ the firms do not share. At $(1, 0; NS)$, both firms invest. At $(X, 1)$ and $(X, 0)$, firm 2 invests.

We know from the analysis of Region 6 that the firms do not share step 1 at $(1, 0)$ if condition (31) holds. This condition is violated in Region 7 and we have

$$\pi^M < \frac{4 (\alpha + r) \pi^D}{2\alpha + r} + \frac{2cr}{2\alpha + r}.$$

Hence, **the firms share step 1 at** $(1, 0)$.

At $(0, 0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2 (0, 0) = \frac{\alpha V_2 (0, 1) + \alpha V_2 (1, 0) - c}{2\alpha + r} = \frac{\alpha V_J (1, 0) - c}{2\alpha + r} = \frac{\alpha V_J (1, 1) - c}{2\alpha + r} > 0$$

where the last equality follows because the firms share at $(1, 0)$. Substituting for $V_J (1, 1) = 2V_2 (1, 1)$ from (7) and simplifying we get

$$\pi^D > \frac{cr}{\alpha} \left(1 + \frac{r}{4\alpha}\right)$$

which holds in Region 7 by assumption. Hence, firm 2 will invest if firm 2 invests. As in Region 6, firm 2 invests at $(X, 0)$. It follows that **both firms invest at** $(0, 0)$.

5.5 Region 8 (monotonicity property holds)

In Region 8, the analysis for the histories $(2, 2)$, $(2, 1; NS)$, $(2, 1)$, $(1, 1)$, $(2, 0; NS)$, $(X, 2)$ and $(X, 1)$ is the same as in Region 6, and we do not repeat it here. At $(2, 2)$ both firms produce output, and at $(X, 2)$ firm 2 produces output. At $(2, 1; NS)$, firm 1 produces output and firm 2 invests. At $(2, 1)$, the firms share. At $(1, 1)$, both firms invest. At $(2, 0; NS)$, firm 1 produces output and firm 2 exits the race. At $(X, 1)$, firm 2 invests.

We know from the analysis of Region 6 that the firms do not share step 1 at $(2, 0)$ if condition (28) holds. This condition is violated in Region 7 and we have

$$\pi^M < 2\pi^D.$$

Hence, **the firms share step 1 at** $(2, 0)$.

At $(1, 0; NS)$, firm 2 invests if

$$V_2(1, 0; NS) = \frac{\alpha V_2(1, 1) + \alpha V_2(2, 0) - c}{2\alpha + r} > 0.$$

At $(2, 0)$, the firms share step 1. However, since the lagging firm exits at $(2, 0; NS)$ and it has no bargaining power, $V_2(2, 0) = 0$. Hence, the investment condition simplifies to

$$V_2(1, 1) > \frac{c}{\alpha}.$$

Substituting for $V_2(1, 1)$ from (7), we get

$$\pi^D > \frac{cr}{\alpha}\left(\frac{3}{2} + \frac{r}{2\alpha}\right). \tag{32}$$

This condition holds in Region 8 and **the lagging firm stays in the race at** $(1, 0; NS)$. It is straightforward to show that **the leading firm also stays in the race at** $(1, 0; NS)$**.**

To see whether the firms share step 1 at $(1, 0)$, we compare joint profits under sharing with joint profits under no sharing. If the firms share, the game reaches the history $(1, 1)$. Hence, joint profits under sharing are $V_J(1, 1)$. Joint profits under no sharing are

$$V_J(1, 0; NS) = \frac{\alpha V_J(2, 0) + \alpha V_J(1, 1) - 2c}{2\alpha + r}.$$

Since the firms share at $(2, 0)$ and $(2, 1)$, we get

$$V_J(1, 0; NS) = \frac{2\alpha\widetilde{\pi}^D + \alpha V_J(1, 1) - 2c}{2\alpha + r}.$$

We have $S \succ NS \iff$

$$(2\alpha + r) V_J (1,1) > 2\alpha \widetilde{\pi}^D + \alpha V_J (1,1) - 2c.$$

Substituting for $V_J(1,1) = 2V_2(1,1)$ from (7) and simplifying, we get

$$\begin{aligned}(\alpha + r)\left(4\alpha\widetilde{\pi}^D - 2c\right) &> (2\alpha + r)\left(2\alpha\widetilde{\pi}^D - 2c\right) \\ \pi^D &> -c,\end{aligned}$$

which is trivially true. Hence, **the firms share at** $(1,0)$ **in Region 8**.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0) + \alpha V_2(0,1) - c}{2\alpha + r} = \frac{\alpha V_J(1,0) - c}{2\alpha + r} = \frac{\alpha V_J(1,1) - c}{2\alpha + r} > 0,$$

where the last equality follows because the firms share at the history $(1,0)$. After substituting for $V_J(1,1)$ we get

$$\pi^D > \frac{cr}{\alpha}\left(1 + \frac{r}{4\alpha}\right).$$

This condition holds in Region 8 by assumption. Hence, each firm invests at $(0,0)$ if the other firm does.

Assuming firm 1 does not invest, firm 2 will invest at $(X,0)$ if $V_2(X,0) > 0$. From the analysis of Region 1, $V_2(X,0) > 0$ if (13) holds. That is, $V_2(X,0) > 0$ if

$$\pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right),$$

which may or may not hold in Region 8. Hence, **firm 2 may or may not invest at** $(X,0)$**.**

We divide Region 8 into two subregions depending on whether condition (13) holds. **In subregion 8a,** (13) holds and **both firms invest at** $(0,0)$. **In subregion 8b**, (13) does not hold and **there are two continuation equilibria at** $(0,0)$**. In one equilibrium, both firms invest and in the other one, neither firm invests.**

5.6 Region 9 (monotonicity property holds)

In Region 9, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(2,1)$, $(1,1)$, $(2,0;NS)$, $(2,0)$, $(X,2)$ and $(X,1)$ is the same as in Region 8 and we do not repeat it here. At $(2,2)$, both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 invests. At $(2,1)$, the firms share. At $(1,1)$, both firms invest. At $(2,0;NS)$, firm 1 produces output and firm 2 exits the race. At $(2,0)$ the firms share step 1. At $(X,1)$, firm 2 invests.

We know from the analysis in Region 8 that at $(1,0;NS)$, firm 2 stays in the race if condition (32) holds. In Region 9 this condition is violated and we have

$$\pi^D < \frac{cr}{\alpha}\left(\frac{3}{2}+\frac{r}{2\alpha}\right).$$

Hence, **firm 2 exits at** $(1,0;NS)$. The condition for the leading firm to stay in the race at $(1,0;NS)$ is the same as the condition for firm 2 it to invest at $(X,1)$. This condition is given in (9) and it holds in Region 9. Hence, **the leading firm invests at** $(1,0;NS)$**.**

To see whether the firms share step 1 at $(1,0)$, we compare joint profits under sharing with joint profits under no sharing. If the firms share, the game reaches the history $(1,1)$. Hence, joint profits under sharing are

$$V_J(1,1) = \frac{\alpha V_J(2,1)+\alpha V_J(1,2)-2c}{2\alpha+r} = \frac{2\alpha V_J(2,2)-2c}{2\alpha+r} = \frac{4\alpha\widetilde{\pi}^D-2c}{2\alpha+r}, \tag{33}$$

where the last two equalities follow because the firms share step 2 at $(2,1)$ and $(1,2)$. Joint profits under no sharing are

$$V_J(1,0;NS) = \frac{\alpha\widetilde{\pi}^M - c}{\alpha+r}$$

since the lagging firm exits at $(1,0;NS)$. We have $S \succ NS \iff$

$$\begin{aligned}(\alpha+r)\left(4\alpha\widetilde{\pi}^D-2c\right) &> (2\alpha+r)\left(\alpha\widetilde{\pi}^M-c\right)\\ 4(\alpha+r)\pi^D-\frac{cr^2}{\alpha} &> \pi^M(2\alpha+r).\end{aligned}$$

In Region 9, we have

$$2\pi^D > \pi^M > \pi^D$$

and

$$\pi^D > \frac{cr}{\alpha}.$$

Using these conditions, it is straightforward to show that the condition for sharing holds at $(1,0)$. Hence, in Region 9, **the firms share at the history** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0)+\alpha V_2(0,1)-c}{2\alpha+r} = \frac{\alpha V_J(1,1)-c}{2\alpha+r} > 0,$$

where the last equality follows because the firms share at the history $(1,0)$. After substituting for $V_J(1,1) = 2V_2(1,1)$ from (7), we get

$$\pi^D > \frac{cr}{\alpha}\left(1+\frac{r}{4\alpha}\right). \tag{34}$$

This condition may or may not hold in Region 9.

Assuming firm 1 does not invest, firm 2 will invest at $(X, 0)$ if $V_2(X, 0) > 0$. From the analysis of Region 1, $V_2(X, 0) > 0$ if (13) holds. That is, $V_2(X, 0) > 0$ if

$$\pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right),$$

which may or may not hold in Region 9. Hence, **firm 2 may or may not invest at** $(X, 0)$.

Conditions (34) and (13) divide Region 9 into **three subregions** depending on whether they hold.[5] **In subregion 9a**, both conditions hold and **both firms invest at** $(0, 0)$. **In subregion 9b**, only (34) holds and there are two continuation equilibria at $(0, 0)$. **In one equilibrium, both firms invest and in the other one, neither firm invests. In subregion 9c,** neither condition holds and **neither firm invests at** $(0, 0)$**.**

5.7 Region 10 (monotonicity property holds)

In Region 10, the analysis for the histories $(2, 2)$, $(2, 1; NS)$, $(2, 1)$, $(1, 1)$, $(2, 0; NS)$, $(2, 0)$, $(X, 2)$ and $(X, 1)$ is the same as in Region 6, and we do not repeat it here. At $(2, 2)$ both firms produce output, and at $(X, 2)$ firm 2 produces output. At $(2, 1; NS)$, firm 1 produces output and firm 2 invests. The firms share at $(2, 1)$, and both firms invest at $(1, 1)$. At $(2, 0; NS)$, firm 1 produces output and firm 2 exits the race. At $(2, 0)$ the firms do not share. At $(X, 1)$, firm 2 invests.

We know from the analysis of Region 6 that the lagging firm stays in the race at $(1, 0; NS)$ if condition (29) holds. This condition is violated in Region 10 and we have

$$\pi^D < \frac{cr}{\alpha}\left(\frac{3}{2} + \frac{r}{2\alpha}\right). \tag{35}$$

Hence, **firm 2 does not invest at** $(1, 0; NS)$. The condition for the leading firm to stay in the race at $(1, 0; NS)$ is the same as the condition for firm 2 it to invest at $(X, 1)$. This condition is given in (9) and it holds in Region 10. Hence, **the leading firm invests at** $(1, 0; NS)$**.**

To see whether the firms share step 1 at $(1, 0)$, we compare joint profits under sharing with joint profits under no sharing. Using (7), joint profits under sharing are

$$V_J(1, 1) = 2V_2(1, 1) = \frac{4\alpha\widetilde{\pi}^D - 2c}{2\alpha + r}$$

Joint profits under no sharing are

[5] There are only 3 subregions (rather than 4 subregions) because the boundary lines given by the conditions intersect either on the border or outside of Region 9.

$$V_J(1,0;NS) = \frac{\alpha\widetilde{\pi}^M - c}{\alpha + r}$$

because the lagging firm exits. Hence, we have $S \succ NS \Longleftrightarrow$

$$\begin{aligned} (\alpha + r)\left(4\alpha\widetilde{\pi}^D - 2c\right) &> (2\alpha + r)\left(\alpha\widetilde{\pi}^M - c\right) \\ \frac{4(\alpha + r)\pi^D}{(2\alpha + r)} - \frac{cr^2}{\alpha(2\alpha + r)} &> \pi^M. \end{aligned} \quad (36)$$

This condition holds in Region 10 and **the firms share step 1 at** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0) + \alpha V_2(0,1) - c}{2\alpha + r} = \frac{\alpha V_J(1,0) - c}{2\alpha + r} > 0.$$

Since the firms share at $(1,0)$, we get

$$\frac{\alpha V_J(1,1) - c}{2\alpha + r} > 0.$$

Substituting for $V_J(1,1) = 2V_2(1,1)$ from (7) and simplifying we get

$$\pi^D > \frac{cr}{\alpha}\left(1 + \frac{r}{4\alpha}\right). \quad (37)$$

This condition may or may not hold in Region 10.

Assuming firm 1 does not invest, firm 2 will invest at $(X,0)$ if $V_2(X,0) > 0$. From the analysis of Region 1, $V_2(X,0) > 0$ if (13) holds. That is, $V_2(X,0) > 0$ if

$$\pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right),$$

which may or may not hold in Region 10. Hence, **firm 2 may or may not invest at** $(X,0)$.

Conditions (37) and (13) divide Region 10 into **three subregions** depending on whether they hold.[6] **In subregion 10a**, both conditions hold and **both firms invest at** $(0,0)$. **In subregion 10b**, only (37) holds and there are two continuation equilibria at $(0,0)$. **In one equilibrium, both firms invest** and **in the other one, neither firm invests. In subregion 10c**, neither condition holds and **neither firm invests at** $(0,0)$.

[6] There are only 3 subregions (rather than 4 subregions) because the boundary lines given by the conditions intersect either on the border or outside of Region 10.

5.8 Region 11 (monotonicity property fails off the equilibrium path)

In Region 11, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(2,1)$, $(1,1)$, $(2,0;NS)$, $(2,0)$, $(1,0;NS)$, $(X,2)$ and $(X,1)$ is the same as in Region 10, and we do not repeat it here. At $(2,2)$ both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 invests. The firms share at $(2,1)$, and both firms invest at $(1,1)$. At $(2,0;NS)$, firm 1 produces output and firm 2 exits the race. At $(2,0)$ the firms do not share. At $(1,0;NS)$, firm 2 exits the race and firm 1 invests. At $(X,1)$, firm 2 invests.[7]

We know from the analysis of Region 10 that the firms share at $(1,0)$ if condition (36) holds. This condition is violated in Region 11 and we have

$$\frac{4(\alpha+r)\pi^D}{(2\alpha+r)} - \frac{cr^2}{\alpha(2\alpha+r)} > \pi^M.$$

Hence, **the firms do not share step 1 at** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0) + \alpha V_2(0,1) - c}{2\alpha+r} = \frac{\alpha V_J(1,0) - c}{2\alpha+r} > 0. \tag{38}$$

Since the firms do not share $(1,0)$ and the lagging firm drops out at $(1,0;NS)$, we have

$$V_J(1,0) = \frac{\alpha\widetilde{\pi}^M - c}{\alpha+r}.$$

Substituting for $V_J(1,0)$ in (38) and simplifying we get

$$\pi^M > \frac{cr}{\alpha}\left(2+\frac{r}{\alpha}\right). \tag{39}$$

This condition may or may not hold in Region 11.

Assuming firm 1 does not invest, firm 2 will invest at $(X,0)$ if $V_2(X,0) > 0$. From the analysis of Region 1, $V_2(X,0) > 0$ if (13) holds. That is, $V_2(X,0) > 0$ if

$$\pi^M > \frac{cr}{\alpha}\left(2+\frac{r}{\alpha}\right),$$

which is the same condition as (39). Hence, **firm 2 may or may not invest at** $(X,0)$.

Conditions (39) divides Region 11 into **two subregions**. **In subregion 11a, condition (39) holds and both firms invest at** $(0,0)$**. In subregion 11b, condition (39) does not hold and neither firm invests at** $(0,0)$**.**

[7] It is straightforward to show that Region 11 is non-empty for some values of π^M and π^D if and only if $\frac{r}{\alpha} < 2$. An explanation is available on request.

5.9 Region 12 (monotonicity property holds)

In Region 12, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(2,1)$, $(1,1)$, $(2,0;NS)$,$(2,0)$, $(X,2)$ and $(X,1)$ is the same as in Region 4, and we do not repeat it here. At $(2,2)$ both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 invests. At $(2,1)$, the firms do not share step 2. At $(1,1)$, both firms invest. At $(2,0;NS)$, firm 1 produces output and firm 2 exits the race. At $(2,0)$, the firms do not share. At $(X,1)$, firm 2 invests. [8]

From the analysis of Region 4, we know that the lagging firm stays in the race at $(1,0;NS)$ if condition (23) holds. In Region 12, this condition is violated and we have

$$\pi^M + 2\alpha\widetilde{\pi}^D < c\left(2+\frac{r}{\alpha}\right)^2.$$

Hence, **the lagging firm exits at** $(1,0;NS)$. The condition for the leading firm to stay in the race at $(1,0;NS)$ is the same as the condition for firm 2 it to invest at $(X,1)$. This condition is given in (9) and it holds in Region 12. Hence, **the leading firm invests at** $(1,0;NS)$**.**

To see whether the firms share step 1 at $(1,0)$, we compare joint profits under sharing with joint profits under no sharing. Joint profits under sharing are $V_J(1,1)$. Joint profits under no sharing are

$$V_J(1,0;NS) = \frac{\alpha\widetilde{\pi}^M - c}{\alpha + r}$$

because the lagging firm exits. Hence, we have $S \succ NS \iff$

$$(\alpha + r)\, V_J(1,1) > \left(\alpha\widetilde{\pi}^M - c\right).$$

Substituting for $V_J(1,1) = 2V_2(1,1)$ from (15) and simplifying we get

$$4\alpha\pi^D > (2\alpha - r)\,\pi^M + cr\left(2+\frac{r}{\alpha}\right). \tag{40}$$

The condition holds in Region 12 by assumption. Hence, **the firms share step 1 at** $(1,0)$**.**

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0) + \alpha V_2(0,1) - c}{2\alpha + r} = \frac{\alpha V_J(1,0) - c}{2\alpha + r} > 0.$$

Since the firms share at $(1,0)$, we get

$$\frac{\alpha V_J(1,1) - c}{2\alpha + r} > 0.$$

[8] Region 12 is non-empty for some values of π^M and π^D if and only if $\frac{r}{\alpha} > 1$. An explanation is available on request.

Substituting for $V_J(1,1) = 2V_2(1,1)$ from (15) and simplifying we get

$$\pi^M + 2\alpha\widetilde{\pi}^D > c\left(2 + \frac{r}{\alpha}\right)\left(\frac{3}{2} + \frac{r}{2\alpha}\right). \qquad (41)$$

This condition may or may not hold in Region 12.

Assuming firm 1 does not invest, firm 2 will invest at $(X,0)$ if $V_2(X,0) > 0$. From the analysis of Region 1, $V_2(X,0) > 0$ if (13) holds. That is, $V_2(X,0) > 0$ if

$$\pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right),$$

which may or may not hold in Region 12. Hence, **firm 2 may or may not invest at** $(X,0)$.

Conditions (41) and (13) divide Region 12 into **three subregions** depending on whether they hold. **In subregion 12a,** both conditions hold and **both firms invest at** $(0,0)$**. In subregion 12b,** only (41) holds and **there are two continuation equilibria at** $(0,0)$**. In one equilibrium, both firms invest and in the other one, neither firm invests. In subregion 12c,** neither condition holds and **neither firm invests at** $(0,0)$**.**

5.10 Region 13 (monotonicity property holds)

In Region 13, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(2,1)$, $(1,1)$, $(2,0;NS)$, $(2,0)$, $(1,0;NS)$, $(X,2)$ and $(X,1)$ is the same as in Region 12 and we do not repeat it here. At $(2,2)$ both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 invests. At $(2,1)$, the firms do not share step 2. At $(1,1)$, both firms invest. At $(2,0;NS)$, firm 1 produces output and firm 2 exits the race. At $(2,0)$, the firms do not share. At $(1,0;NS)$, firm 1 invests and firm 2 exits the race. At $(X,1)$, firm 2 invests.[9]

We know from the analysis of Region 12 that the firms share at $(1,0)$ if condition (40) holds. This condition is violated in Region 13 and we have

$$4\alpha\pi^D < (2\alpha - r)\,\pi^M + cr\left(2 + \frac{r}{\alpha}\right).$$

Hence, **the firms do not share step 1 at** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0) + \alpha V_2(0,1) - c}{2\alpha + r} = \frac{\alpha V_J(1,0) - c}{2\alpha + r} > 0. \qquad (42)$$

[9] Region 13 is non-empty for some values of π^M and π^D if and only if $\frac{r}{\alpha} < 2$. An explanation is available on request.

Since the firms do not share $(1,0)$ and the lagging firm drops out at $(1,0;NS)$, we have

$$V_J(1,0) = \frac{\alpha\widetilde{\pi}^M - c}{\alpha + r}.$$

Substituting for $V_J(1,0)$ in (42) and simplifying we get

$$\pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right). \tag{43}$$

This condition may or may not hold in Region 13.

Assuming firm 1 does not invest, firm 2 will invest at $(X,0)$ if $V_2(X,0) > 0$. From the analysis of Region 1, $V_2(X,0) > 0$ if (13) holds. That is, $V_2(X,0) > 0$ if

$$\pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right),$$

which is the same condition as (43). Hence, **firm 2 may or may not invest at** $(X,0)$.

Condition (43) divides Region 13 into two subregions. **In region 13a, condition (43) holds and both firms invest at** $(0,0)$**. In region 13b, (43) does not hold and neither firm invests at** $(0,0)$**.**

5.11 Region 14 (monotonicity property holds)

In Region 14, the analysis of the histories $(2,2)$ and $(X,2)$ is the same as in Region 1, and we do not repeat it here. At $(2,2)$ both firms produce output, and at $(X,2)$ firm 2 produces output.

At $(2,1;NS)$**, firm 1 produces output.** Firm 2 invests if condition (3) holds. This condition is violated in Region 14 and we have

$$\widetilde{\pi}^D < \frac{c}{\alpha}.$$

Hence, **firm 2 does not invest at** $(2,1;NS)$**.**

To see whether the firms share step 2 at $(2,1)$, we compare joint profits under sharing with joint profits under no sharing. Joint profits under sharing are $V_J(2,2) = 2\widetilde{\pi}^D$. Joint profits under no sharing are

$$V_J(2,1;NS) = V_1(2,1;NS) = \widetilde{\pi}^M \tag{44}$$

since the lagging firm exits. Hence, we have

$$NS \succ S \Longleftrightarrow \pi^M > 2\pi^D.$$

This condition holds in Region 14 and **the firms do not share step 2 at** $(2,1)$.

At $(1,1)$, assuming firm 1 invests, firm 2 also invests if

$$V_2(1,1) = \frac{\alpha V_2(1,2) + \alpha V_2(2,1) - c}{2\alpha + r} = \frac{\alpha V_J(2,1) - c}{2\alpha + r} > 0.$$

Since the firms do not share at the history $(2,1)$ we can substitute for $V_J(2,1) = \widetilde{\pi}^M$ using (44) to get

$$V_2(1,1) = \frac{\alpha\widetilde{\pi}^M - c}{2\alpha + r} > 0 \tag{45}$$

or

$$\pi^M > \frac{cr}{\alpha}. \tag{46}$$

This condition holds in Region 14 by assumption so that firm 2 invests if firm 1 does.

If firm 1 does not invest at $(1,1)$, the new history is $(X,1)$. Firm 2 will invest at $(X,1)$ if

$$V_2(X,1) = \frac{\alpha\widetilde{\pi}^M - c}{\alpha + r} > 0$$

This is the same condition as above, so that **firm 2 invests at** $(X,1)$. It follows that **at** $(1,1)$**, there is a unique continuation equilibrium such that both firms invest .**

At $(2,0;NS)$**, firm 1 produces output.** Firm 2 invests if

$$V_2(2,0;NS) = \frac{\alpha V_2(2,1) - c}{\alpha + r} > 0.$$

However, since the firms do not share at $(2,1)$ and the lagging firm exits at $(2,1;NS)$, this condition never holds. Hence, **the lagging firm exits at** $(2,0;NS)$ also.

The firms share step 1 at $(2,0)$ if joint profits under sharing exceed joint profits under no sharing. If there is sharing, the game moves to the history $(2,1)$. The firms do not share step 2 at $(2,1)$ and the lagging firm exits. Hence, joint profits under sharing at the history $(2,0)$ are given by $V_J(2,1) = \widetilde{\pi}^M$. Joint profits under no sharing are given by $V_J(2,0;NS) = \widetilde{\pi}^M$ since the lagging firm exits at $(2,0;NS)$. Hence, **we get two continuation equilibria at** $(2,0)$. The equilibria are payoff-equivalent. **In both cases, the lagging firm exits and the leading firm earns** $\widetilde{\pi}^M$.

At $(1,0;NS)$, the lagging firm invests if

$$V_2(1,0;NS) = \frac{\alpha V_2(1,1) + \alpha V_2(2,0) - c}{2\alpha + r} > 0.$$

Since $V_2(2,0) = 0$, this condition simplifies to

$$V_2(1,1) > \frac{c}{\alpha}.$$

Substituting for $V_2(1,1)$ using (45), we get

$$\pi^M > \frac{cr}{\alpha}\left(3+\frac{r}{\alpha}\right). \tag{47}$$

In Region 14, this condition holds and **the lagging firm stays in the race at** $(1,0;NS)$. It is straightforward to show that **the leading firm also stays in at** $(1,0;NS)$**.**

To see whether the firms share step 1 at $(1,0)$, we compare joint profits under sharing with joint profits under no sharing. Joint profits under sharing are $V_J(1,1)$. Joint profits under no sharing are

$$V_J(1,0;NS) = \frac{\alpha V_J(2,0)+\alpha V_J(1,1)-2c}{2\alpha+r} = \frac{\alpha\widetilde{\pi}^M+\alpha V_J(1,1)-2c}{2\alpha+r}$$

since $V_J(2,0)=\widetilde{\pi}^M$ as explained above.

We have $S \succ NS \iff$

$$(\alpha+r)V_J(1,1) > \alpha\widetilde{\pi}^M - 2c.$$

Substituting for $V_J(1,1)=2V_2(1,1)$ from (45) and simplifying we get

$$\begin{aligned}(\alpha+r)\left(\frac{2\alpha\widetilde{\pi}^M-2c}{2\alpha+r}\right) &> \alpha\widetilde{\pi}^M-2c\\ 2(\alpha+r)\left(\alpha\widetilde{\pi}^M-c\right) &> (2\alpha+r)\left(\alpha\widetilde{\pi}^M-2c\right)\end{aligned}$$

which clearly holds. Hence, in Region 14, **the firms share at the history** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0)+\alpha V_2(0,1)-c}{2\alpha+r} = \frac{\alpha V_J(1,1)-c}{2\alpha+r} > 0$$

Substituting for $V_J(1,1)=2V_2(1,1)$ using (45) we get

$$\pi^M > \frac{cr}{\alpha}\left(2+\frac{r}{2\alpha}\right).$$

This condition holds in Region 14. Hence, each firm invests at $(0,0)$ if the other firm does.

Assuming firm 1 does not invest, firm 2 will invest at $(X,0)$ if $V_2(X,0)>0$. From the analysis of Region 1, $V_2(X,0)>0$ if (13) holds. That is, $V_2(X,0)>0$ if

$$\pi^M > \frac{cr}{\alpha}\left(2+\frac{r}{\alpha}\right),$$

which holds in Region 14 by assumption. Hence, **firm** 2 **invests at** $(X,0)$ **in Region 14**.

We conclude that **in equilibrium, both firms invest at** $(0,0)$ in Region 14.

5.12 Region 15 (monotonicity property holds)

In Region 15, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(2,1)$, $(1,1)$, $(2,0;NS)$, $(2,0)$, $(X,2)$ and $(X,1)$ is the same as in Region 14, and we do not repeat it here. At $(2,2)$ both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 exits the race. At $(2,1)$, the firms do not share step 1, and at $(1,1)$, both firms invest. At $(2,0;NS)$, firm 1 produces output and firm 2 exits the race. At $(2,0)$, the firms may either share or not share step 1, but either way the lagging firm 2 exits the race and firm 1 produces output. At $(X,1)$, firm 2 invests.

We know from the analysis of Region 14 that the lagging firm invests at $(1,0;NS)$ if condition (47) holds. In Region 15 this condition is violated. We have

$$\pi^M < \frac{cr}{\alpha}\left(3+\frac{r}{\alpha}\right)$$

and **the lagging firm exits at** $(1,0;NS)$. The condition for the leading firm to stay in the race at $(1,0;NS)$ is the same as the condition for firm 2 it to invest at $(X,1)$. This condition is given in (9), and it holds in Region 15. Hence, **the leading firm invests at** $(1,0;NS)$**.**

To see whether the firms share step 1 at $(1,0)$, we compare joint profits under sharing with joint profits under no sharing. If the firms share, the game reaches the history $(1,1)$. Hence, joint profits under sharing are

$$V_J(1,1) = \frac{\alpha V_J(2,1) + \alpha V_J(1,2) - 2c}{2\alpha + r} = \frac{2\alpha\widetilde{\pi}^M - 2c}{2\alpha + r} \tag{48}$$

where we substitute for $V_J(2,1) = \widetilde{\pi}^M$ from (44). Joint profits under no sharing are

$$V_J(1,0;NS) = \frac{\alpha\widetilde{\pi}^M - c}{\alpha + r}$$

since the lagging firm exits at the history $(1,0;NS)$. We have $S \succ NS \iff$

$$2(\alpha + r) > (2\alpha + r)$$

which clearly holds. Hence, in Region 15, **the firms share at the history** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_1(1,0) + \alpha V_1(0,1) - c}{2\alpha + r} = \frac{\alpha V_J(1,1) - c}{2\alpha + r} > 0.$$

Substituting for $V_J(1,1)$ from (48) we get

$$\pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{2\alpha}\right). \tag{49}$$

This condition may or may not hold in Region 15.

Assuming firm 1 does not invest, firm 2 will invest at $(X,0)$ if $V_2(X,0) > 0$. From the analysis of Region 1, $V_2(X,0) > 0$ if (13) holds. That is, $V_2(X,0) > 0$ if

$$\pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right),$$

which may or may not hold in Region 15. Hence, **firm 2 may or may not invest at** $(X,0)$.

Conditions (49) and (13) divide Region 15 into **three subregions** depending on whether they hold. **In subregion 15a**, both conditions hold and **both firms invest at** $(0,0)$**. In subregion 15b,** only (49) holds and we have

$$\frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right) > \pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{2\alpha}\right).$$

There are two continuation equilibria at $(0,0)$**. In one equilibrium, both firms invest and in the other one, neither firm invests. In subregion 15c,** neither condition holds and **neither firm invests at** $(0,0)$.

5.13 Region 16 (monotonicity property holds)

In Region 16, the analysis for the histories $(2,2)$, $(2,1;NS)$ and $(X,2)$ is the same as in Region 14 and we do not repeat it here. At $(2,2)$ both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 exits the race.

We know from the analysis in Region 14 that the firms do not share at $(2,1)$ if $\pi^M > 2\pi^D$. This condition is violated in Region 16. We have

$$\pi^M < 2\pi^D \tag{50}$$

and hence **the firms share at** $(2,1)$.

At $(1,1)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(1,1) = \frac{\alpha V_2(1,2) + \alpha V_2(2,1) - c}{2\alpha + r} = \frac{\alpha V_J(2,1) - c}{2\alpha + r} > 0.$$

Since the firms share at $(2,1)$, we get

$$V_2(1,1) = \frac{\alpha V_J(2,2) - c}{2\alpha + r} = \frac{2\alpha\widetilde{\pi}^D - c}{2\alpha + r} > 0 \tag{51}$$

which simplifies to

$$\pi^D > \frac{cr}{2\alpha}. \tag{52}$$

This condition holds in Region 16 by assumption so that firm 2 invests if firm 1 does.

If firm 1 does not invest at $(1,1)$, the new history is $(X,1)$. Firm 2 will invest at $(X,1)$ if

$$V_2(X,1)=\frac{\alpha\widetilde{\pi}^M-c}{\alpha+r}>0$$

or

$$\pi^M>\frac{cr}{\alpha}.$$

This is the same condition as (9) and it holds in Region 16 by assumption. Hence, **firm 2 invests at** $(X,1)$ in Region 16. This implies that **at** $(1,1)$**, both firms invest.**

At $(2,0;NS)$**, firm 1 produces output.** Firm 2 stays in the race under no sharing if

$$V_2(2,0;NS)=\frac{\alpha V_2(2,1)-c}{\alpha+r}>0.$$

We know that since the lagging firm exits at $(2,1;NS)$, $V_2(2,1;NS)=0$. This implies that even if the firms share at $(2,1)$, $V_2(2,1)=V_2(2,1;NS)=0$ since the lagging firm has no bargaining power. Hence, **firm 2 exits at** $(2,0;NS)$.

To see whether the firms share step 1 at $(2,0)$, we compare joint profits under sharing with joint profits under no sharing. Joint profits under sharing step 1 are $V_J(2,1)=2\widetilde{\pi}^D$ since the firms share step 2 at $(2,1)$. Joint profits under no sharing are $V_J(2,0;NS)=\widetilde{\pi}^M$ since the lagging firm drops out at $(2,0;NS)$. We have $S\succ NS\iff 2\pi^D>\pi^M$. This condition holds in Region 16, and hence **the firms share step 1 at** $(2,0)$**.**

The lagging firm stays in at $(1,0;NS)$ if

$$V_2(1,0;NS)=\frac{\alpha V_2(1,1)+\alpha V_2(2,0)-c}{2\alpha+r}>0.$$

Since the lagging firm has no bargaining power, it earns the same payoff at $V_2(2,0)$ as at $V_2(2,0;NS)$. But this payoff is zero, because the lagging firm exists at $(2,0;NS)$. Hence the condition above simplifies to

$$V_2(1,1)>\frac{c}{\alpha}$$

or

$$\pi^D>\frac{cr}{\alpha}\left(\frac{3}{2}+\frac{r}{2}\alpha\right)$$

which fails in Region 16 because $\pi^D<\frac{cr}{\alpha}$. Hence, **the lagging firm exits at** $(1,0;NS)$. Thee condition for the leading firm to stay in the race at $(1,0;NS)$ is the same as the condition for firm 2 to invest at $(X,1)$. As discussed above, this condition holds in Region 16. Hence, **the leading firm stays in the race at** $(1,0;NS)$**.**

To see whether the firms share step 1 at $(1,0)$, we compare joint profits under sharing with joint profits under no sharing. If the firms share, the game reaches the history $(1,1)$.

Hence, joint profits under sharing are

$$V_J(1,1) = \frac{\alpha V_J(2,1) + \alpha V_J(1,2) - 2c}{2\alpha + r} = \frac{2\alpha V_J(2,2) - 2c}{2\alpha + r} = \frac{4\alpha\widetilde{\pi}^D - 2c}{2\alpha + r} \tag{53}$$

since the firms share step 2 at $(2,1)$. Joint profits under no sharing are

$$V_J(1,0;NS) = \frac{\alpha\widetilde{\pi}^M - c}{\alpha + r}$$

since the lagging firm exits at the history $(1,0;NS)$. We have $S \succ NS \iff$

$$\begin{aligned} \left(4\alpha\widetilde{\pi}^D - 2c\right)(\alpha + r) &> \left(\alpha\widetilde{\pi}^M - c\right)(2\alpha + r) \\ 4(\alpha + r)\pi^D - \frac{cr^2}{\alpha} &> \pi^M(2\alpha + r). \end{aligned}$$

In Region 16, by assumption, we know that

$$\frac{2cr}{\alpha} > 2\pi^D > \pi^M.$$

Using this condition, it is straightforward to show that the condition for sharing holds, and hence **the firms share step 1 at** $(1,0)$.

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0) + \alpha V_2(0,1) - c}{2\alpha + r} = \frac{\alpha V_J(1,0) - c}{2\alpha + r} > 0.$$

Since the firms share at $(1,0)$, the condition simplifies to

$$V_J(1,1) > \frac{c}{\alpha}.$$

Substituting for $V_J(1,1)$ from (53) we get

$$\begin{aligned} \frac{4\alpha\widetilde{\pi}^D - 2c}{2\alpha + r} &> \frac{c}{\alpha} \\ \pi^D &> \frac{cr}{\alpha}\left(1 + \frac{r}{4\alpha}\right), \end{aligned}$$

which does not hold in Region 16 since $\pi^D < \frac{cr}{\alpha}$. Hence, if firm 1 invests, firm 2 does not invest at $(0,0)$.

Assuming firm 1 does not invest, firm 2 will invest at $(X,0)$ if $V_2(X,0) > 0$. From the analysis of Region 1, $V_2(X,0) > 0$ if (13) holds. That is, $V_2(X,0) > 0$ if

$$\pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right),$$

which does not hold in Region 16 since

$$\frac{2cr}{\alpha} > 2\pi^D > \pi^M$$

by assumption. Hence, **firm 2 does not invest at** $(X, 0)$.

We conclude that **neither firms invests at** $(0, 0)$.

5.14 Region 17 (monotonicity property can fail off the equilibrium path)

In Region 17, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(2,1)$, $(2,0;NS)$, $(2,0)$ and $(X,2)$ is the same as in Region 16, and we do not repeat it here. At $(2,2)$, both firms produce output. At $(2,1;NS)$, firm 1 produces output and firm 2 exits the race. At $(2,1)$, the firms share step 2. At $(2,0;NS)$, firm 1 produces output and firm 2 exits the race. At $(2,0)$, the firms share step 1. At $(X,2)$, firm 2 produces output.

At $(X,1)$, firm 1 has exited the race. As in Region 16, firm 2 invests if and only if condition (9) holds. In Region 17, the condition fails and **firm 2 exits the race at** $(X,1)$. Given this, **firm 2 also exits the race at** $(X,0)$.

At $(1,1)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(1,1) > 0.$$

As in Region 16, this inequality is the same as condition (52) or

$$\pi^D > \frac{cr}{2\alpha}.$$

Condition (52) holds in Region 17, so firm 2 invests at at $(1,1)$ if firm 1 does.

If firm 1 does not invest at $(1,1)$, the new history is $(X,1)$ and as we showed above, firm 2 exits the race at $(X,1)$. So firm 2 does not invest at $(1,1)$ if firm 1 does not invest.

This implies that **there are two continuation equilibria at** $(1,1)$**. Either both firms invest or neither firm invests.**

The lagging firm stays in at $(1,0;NS)$ if

$$V_2(1,0;NS) = \frac{\alpha V_2(1,1) + \alpha V_2(2,0) - c}{2\alpha + r} > 0.$$

Since the lagging firm has no bargaining power at $(2,0)$, it earns a payoff of $V_2(2,0) = 0$ as in Region 16. Hence, the condition simplifies to

$$V_2(1,1) > \frac{c}{\alpha}.$$

The payoff $V_2(1,1)$ depends on which continuation equilibrium is played.

If neither firm invests at $(1,1)$, then $V_2(1,1) = 0$ and the condition fails. Hence **if both firms exit as** $(1,1)$,**then both firms exit the race at** $(1,0;NS)$.

If both firms invest at $(1,1)$, the payoff $V_2(1,1)$ is the same as in Region 16 and is given in (51). Substituting for $V_2(1,1)$, the condition becomes

$$\pi^D > \frac{cr}{\alpha}\left(\frac{3}{2} + \frac{r}{2}\alpha\right)$$

which fails in region 17 because $\pi^D < \frac{cr}{\alpha}$. Hence, the lagging firm exits at $(1,0;NS)$. The new history is $(1,X)$. Since the leading firm exits at $(1,X)$, the leading firm also exits at $(1,0;NS)$. Hence, **if both firms invest at** $(1,1)$**, then both firms exit the race at** $(1,0;NS)$.

To see whether the firms share step 1 at $(1,0)$, we compare joint profits under sharing with joint profits under no sharing. If the firms do not share, then the joint profits are zero because both firms exit at $(1,0;NS)$, regardless of which continuation equilibrium is played at $(1,1)$.

If the firms share, then the new history is $(1,1)$. If both firms exit the race at $(1,1)$, then the joint profits are zero. The firms are indifferent between sharing and not sharing at $(1,0)$. **Hence, if both firms exit the race at** $(1,1)$, **there are two continuation equilibria at** $(1,0)$. **In one continuation, the firms share at** $(1,0)$**. In the other continuation, the firms do not share at** $(1,0)$**.** If both firms stay in the race at $(1,1)$, then the joint profits $V_J(1,1)$ are the same as in Region 16a and are given in (53). We have $S \succ NS \iff$

$$\frac{4\alpha\widetilde{\pi}^D - 2c}{2\alpha + r} > 0$$

This simplifies to

$$\pi^D > \frac{cr}{2\alpha}$$

which holds in Region 17. Hence, **if both stay in the race at** $(1,1)$, **the firms share step** 1 **at** $(1,0)$**.**

At $(0,0)$, if firm 1 does not invest, then the new history is $(X,1)$ and firm 2 exits the race. This implies that there is an equilibrium at $(0,0)$ such that neither firm invests.

If firm 1 invests at $(0,0)$, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0) + \alpha V_2(0,1) - c}{2\alpha + r} = \frac{\alpha V_J(1,0) - c}{2\alpha + r} > 0. \tag{54}$$

The firms either share or not share at $(1,0)$ depending on which continuation equilibrium is played. If the firms do not share at $(1,0)$, then the new history is $(1,0;NS)$ and both firms exit the race. Hence, $V_J(1,0) = 0$. Substituting into the expression for $V_2(0,0)$, we see that $V_2(0,0) < 0$ so firm 2 exits the race. If instead the firms share at $(1,0)$, then

$V_J(1,0) = V_J(1,1)$. The joint payoffs at $(1,1)$ depend on which continuation equilibrium is played. If both firms exit at $(1,1)$, then $V_J(1,0) = V_J(1,1) = 0$. Using this, we see that $V_2(0,0) < 0$. Hence, firm 2 does not invest at $(0,0)$ if firm 1 invests. If instead both firms invest at $(1,1)$,then $V_J(1,1)$ is the same as in Region 16 and is given in (53). Using this, we can substitute for $V_J(1,0) = V_J(1,1)$ in condition (54). The condition simplifies to

$$\pi^D > \frac{cr}{\alpha}\left(1 + \frac{r}{4\alpha}\right)$$

which does not hold in Region 17 since $\pi^D < \frac{cr}{\alpha}$. Hence, firm 2 does not invest at $(0,0)$ if firm 1 invests.

We conclude that **in every equilibrium in Region 17, neither firm invests at $(0,0)$.**

5.15 Region 18 (monotonicity property can fail off the equilibrium path)

In Region 18, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(2,1)$ and $(X,2)$ is the same as in Region 16, and we do not repeat it here. At $(2,2)$ both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 exits the race. At $(2,1)$, the firms share step 1.

At $(1,1)$, assuming firm 1 invests, firm 2 will also invest if

$$\pi^M > \frac{cr}{\alpha}\left(2 + \frac{r}{\alpha}\right),$$

$$V_2(1,1) = \frac{\alpha V_2(1,2) + \alpha V_2(2,1) - c}{2\alpha + r} = \frac{\alpha V_J(2,1) - c}{2\alpha + r} > 0.$$

Since the firms share at $(2,1)$, this is

$$V_2(1,1) = \frac{\alpha V_J(2,2) - c}{2\alpha + r} = \frac{2\alpha\widetilde{\pi}^D - c}{2\alpha + r} > 0$$

or

$$\pi^D > \frac{cr}{2\alpha}$$

This condition fails in Region 18, and firm 2 does not invest at $(1,1)$ if firm 1 does.

If firm 1 does not invest at $(1,1)$, the new history is $(X,1)$. Firm 2 will invest at $(X,1)$ if

$$V_2(X,1) = \frac{\alpha\widetilde{\pi}^M - c}{\alpha + r} > 0,$$

which simplifies to

$$\pi^M > \frac{cr}{\alpha}. \tag{55}$$

This condition is the same as (9), and it fails in Region 18. Hence, **at** $(X, 1)$**, firm** 2 **does not invest**, and **at** $(1, 1)$, **neither firms invests**.

At $(2, 0; NS)$**, firm 1 produces output.** Firm 2 stays in the race if

$$V_2(2, 0; NS) = \frac{\alpha V_2(2, 1) - c}{\alpha + r} > 0.$$

We know that since the lagging firm exits at $(2, 1; NS)$, $V_2(2, 1; NS) = 0$. This implies that $V_2(2, 1) = V_2(2, 1; NS) = 0$ since the lagging firm has no bargaining power. Hence, **the lagging firm exits at** $(2, 0; NS)$.

To see whether the firms share step 1 at $(2, 0)$, we compare joint profits under sharing with joint profits under no sharing. Joint profits under sharing step 1 are $V_J(2, 1) = 2\widetilde{\pi}^D$ since the firms share step 2 at $(2, 1)$. Joint profits under no sharing are $V_J(2, 0; NS) = \widetilde{\pi}^M$ since the lagging firm drops out at $(2, 0; NS)$. We have $S \succ NS \iff 2\pi^D > \pi^M$. This condition holds in Region 18. Hence, **the firms share step** 1 **at** $(2, 0)$.

The lagging firm stays in the race at $(1, 0; NS)$ if

$$V_2(1, 0; NS) = \frac{\alpha V_2(1, 1) + \alpha V_2(2, 0) - c}{2\alpha + r} > 0.$$

Since neither firm invests at $(1, 1)$ and the lagging firm exits at $(2, 0; NS)$, we have $V_2(1, 1) = 0$ and $V_2(2, 0) = V_2(2, 0; NS) = 0$. Hence, **the lagging firm exits the race at** $(1, 0; NS)$**.** The condition for the leading firm to stay in the race at $(1, 0; NS)$ is the same as the condition for firm 2 to invest at $(X, 1)$. As discussed above, the condition does not hold in Region 18. Hence, **the leading firm does not invest at** $(1, 0; NS)$.

At $(1, 0)$, joint profits under sharing and not sharing are both zero since the firms choose not to stay in the race at $(1, 1)$ and $(1, 0; NS)$. Hence, **there are two continuation equilibria at** $(1, 0)$. These equilibria are payoff-equivalent. **In the first continuation equilibrium, the firms do not share step** 1 **and both exit the race. In the second continuation equilibrium, the firms do share step 1 and then both exit the race.**

At $(0, 0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0, 0) = \frac{\alpha V_2(1, 0) + \alpha V_2(0, 1) - c}{2\alpha + r} = \frac{\alpha V_J(1, 0) - c}{2\alpha + r} > 0.$$

Since $V_J(1, 0) = 0$, if firm 1 invests, firm 2 does not invest at $(0, 0)$.

Assuming firm 1 does not invest, firm 2 invests at $(X, 0)$ if

$$V_2(X, 0) = \frac{aV_2(X, 1) - c}{\alpha + r} > 0.$$

Since firm 2 does not invest at $(X, 1)$, $V_2(X, 1) = 0$. Hence, **firm 2 does not invest at** $(X, 0)$.

We conclude that **neither firms invests at** $(0, 0)$ **in Region 18**.

5.16 Region 19 (monotonicity property holds)

In Region 19, the analysis for the histories $(2,2)$, $(2,1;NS)$, $(2,1)$ and $(X,2)$ is the same as in Region 15, and we do not repeat it here. At $(2,2)$, both firms produce output, and at $(X,2)$ firm 2 produces output. At $(2,1;NS)$, firm 1 produces output and firm 2 exits the race. At $(2,1)$, the firms do not share step 1.

At $(1,1)$, we know from the analysis of Region 15 that whether or not firm 1 invests, firm 2 invests if

$$\pi^M > \frac{cr}{\alpha}.$$

This condition is the same as (9), and it fails in Region 19. Hence **firm 2 does not invest at** $(X,1)$ and **neither firm invests at** $(1,1)$.

At $(2,0;NS)$, firm 1 produces output. The lagging firm 2 invests if

$$V_2\left(2,0;NS\right) = \frac{\alpha V_2\left(2,1\right) - c}{\alpha + r} > 0.$$

However, since the firms do not share at $(2,1)$ and the lagging firm exits at $(2,1;NS)$, we have $V_2(2,1) = 0$ and this condition never holds. Hence, **the lagging firm exits at** $(2,0;NS)$.

The firms share step 1 at $(2,0)$ if joint profits under sharing exceed joint profits under no sharing. If there is sharing, the game moves to the history $(2,1)$. The firms do not share step 2 at $(2,1)$ and the lagging firm exits. Hence, joint profits under sharing at the history $(2,0)$ are given by $V_J(2,1) = \widetilde{\pi}^M$. Joint profits under no sharing are given by $V_J(2,0;NS) = \widetilde{\pi}^M$ since the lagging firm exits at $(2,0;NS)$. Hence, we get **two continuation equilibria at** $(2,0)$**. The firms either share or don't share step 1.** The lagging firm then exits the race.

At $(1,0;NS)$, the lagging firm invests if

$$V_2\left(1,0;NS\right) = \frac{\alpha V_2\left(1,1\right) + \alpha V_2\left(2,0\right) - c}{2\alpha + r} > 0.$$

Since neither firm invests at $(1,1)$ and the lagging firm gets zero at $(2,0)$, we have $V_2(1,1) = 0$ and $V_2(2,0) = 0$. This implies that **the lagging firm exits the race at** $(1,0;NS)$.

The condition for the leading firm to stay in the race at $(1,0;NS)$ is the same as the condition for firm 2 to invest at $(X,1)$. As discussed above, the condition does not hold in Region 19. Hence, **the leading firm exits the race at** $(1,0;NS)$.

At $(1,0)$, joint profits under sharing and not sharing are both zero since the firms choose not to stay in the race at $(1,1)$ and $(1,0)$. Hence, **there are two continuation equilibria at** $(1,0)$**. The firms either share or do not share step 1. Both firms then exit the race.**

At $(0,0)$, assuming firm 1 invests, firm 2 will also invest if

$$V_2(0,0) = \frac{\alpha V_2(1,0) + \alpha V_2(0,1) - c}{2\alpha + r} = \frac{\alpha V_J(1,1) - c}{2\alpha + r} > 0$$

Since $V_J(1,1) = 0$, if firm 1 invests, firm 2 does not invest at $(0,0)$.

Assuming firm 1 does not invest, firm 2 invests at $(X,0)$ if

$$V_2(X,0) = \frac{aV_2(X,1) - c}{\alpha + r} > 0.$$

Since firm 2 does not invest at $(X,1)$, $V_2(X,1) = 0$. Hence, **firm 2 does not invest at** $(X,0)$.

We conclude that **neither firms invests at** $(0,0)$ **in Region 19**.